STRONG SYMBOLIC DYNAMICS FOR GEODESIC FLOW ON CAT(-1) SPACES AND OTHER METRIC ANOSOV FLOWS

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Abstract. We prove that the geodesic flow on a compact, locally CAT(-1) metric space can be coded by a suspension flow over an irreducible shift of finite type with Hölder roof function. This is achieved by showing that the geodesic flow is a metric Anosov flow, and obtaining Hölder regularity of return times for a special class of geometrically constructed local cross-sections to the flow. We obtain a number of strong results on the dynamics of the flow with respect to equilibrium measures for Hölder potentials. In particular, we prove that the Bowen-Margulis measure is Bernoulli except for the exceptional case that all closed orbit periods are integer multiples of a common constant. We show that our techniques also extend to the geodesic flow associated to a projective Anosov representation [BCLS15], which verifies that the full power of symbolic dynamics is available in that setting.

1. Introduction

A *metric Anosov flow*, or *Smale flow*, is a topological flow equipped with a continuous bracket operation which abstracts the local product structure of uniform hyperbolic flows. Examples of metric Anosov flows include Axiom A flows, suspension flows over shifts of finite type with no regularity beyond continuity required on the roof function, and the flows associated to projective Anosov representations studied by Bridgeman, Canary, Labourie and Sambarino [BCLS15, BCS17].

We say that a system has a *Markov coding* if there is a finite-to-one surjective semi-conjugacy π with a suspension flow over an irreducible shift of finite type on a finite alphabet. However, for this symbolic description to be useful, it is also required that the roof function and the map π can be taken to be Hölder. We call this a *strong Markov coding*. Pollicott showed that Bowen's construction of symbolic dynamics for Axiom A flows can be extended to the metric Anosov setting [Pol87] to provide a Markov coding. However, no criteria for obtaining a *strong* Markov coding, which is necessary for most dynamical applications, were suggested. Examples of metric Anosov flows which do not have a strong Markov coding are easily provided by the class of suspension flows over a shift of finite type where the roof function is continuous but not Hölder. In this paper, we give a method for obtaining the strong Markov coding for some systems of interest via the metric Anosov flow machinery.

Our primary motivation for this analysis is to gain a more complete dynamical picture for the geodesic flow on a compact, locally CAT$(-\kappa)$ metric space X, where $\kappa > 0$, which is a generalization of the geodesic flow on a closed Riemannian manifold

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of negative curvature. In the Riemannian case, the geodesic flow is Anosov, so the system has a strong Markov coding by Bowen's results [Bow73]. We show that this extends to the CAT$(-\kappa)$ case. Previous dynamical results in this area are mainly based on analysis of the boundary at infinity via the Patterson–Sullivan construction. This has yielded many results for the Bowen-Margulis measure [Rob03], which have been recently extended to a natural class of equilibrium states [BAPP16]. A weak form of symbolic dynamics for geodesic flows is due to Gromov [Gro87], and expanded upon by Coornaert and Papadopoulos [CP12]. This uses topological arguments to give an orbit semi-equivalence with a suspension over a subshift of finite type. A priori, orbit semi-equivalence is too weak a relationship to preserve any interesting dynamical properties [GM10, KT17], and it is not known how to improve this construction of symbolic dynamics to a semi-conjugacy. In [CLT16], we used this weak symbolic description to prove that these geodesic flows are expansive flows with the weak specification property, and explored the consequences of this characterization. However, neither the boundary at infinity techniques, nor techniques based on the specification property are suitable for proving finer dynamical results such as the Bernoulli property. Once the strong Markov coding is established, a treasure trove of results from the literature can be applied. We collect some of these as they apply for geodesic flow on a CAT(-1) spaces as Corollary C. The Bernoulli property in particular is an application that is out of reach of the previous techniques available in this setting.

Our first step is to formulate verifiable criteria for a metric Anosov flow to admit a strong Markov coding. In the following statement, the *pre-Markov proper families at scale* α, which are formally introduced in Definition 3.7, are families of sections to the flow $\mathcal{B} = \{B_i\}$, $\mathcal{D} = \{D_i\}$, $B_i \subset D_i$ with finite cardinality and diameter less than α satisfying certain nice basic topological and dynamical properties. These families were originally introduced by Bowen and are the starting point for his construction of symbolic dynamics for flows.

Theorem A. *Let* $\{\phi_t\}$ *be a Hölder continuous metric Anosov flow. Suppose that there exists a pre-Markov proper family* $(\mathcal{B}, \mathcal{D})$ *satisfying:*

(1) *the return time function* $r(y)$ *for* $\mathcal{B}$ *is Hölder where it is continuous (i.e., on each* $B_i \cap H^{-1}(B_j)$ *where* H *is the Poincaré first-return map);*
(2) *the projection maps along the flow* $\mathrm{Proj}_{B_i} : B_i \times [-\alpha, \alpha] \to B_i$ *are Hölder, where* $\alpha > 0$ *is the scale for the pre-Markov proper family.*

Then the flow has a strong Markov coding.

We then verify these criteria in our setting, obtaining the following application, which is our main result.

Theorem B. *The geodesic flow on a compact, locally* CAT$(-\kappa)$ *space, where* $\kappa > 0$*, has a strong Markov coding.*

We prove Theorem B by giving a geometric construction of a 'special' pre-Markov proper family $(\mathcal{B}, \mathcal{D})$ for the geodesic flow on a compact, locally CAT$(-\kappa)$ space. The sections are defined in terms of Busemann functions, which are well known to be Lipschitz. We then use the regularity of the Busemann functions to establish (1) and (2) of Theorem A for the family $(\mathcal{B}, \mathcal{D})$, thus establishing Theorem B.

Our second main application is to use similar techniques to study the flow associated to a projective Anosov representation, which is another important example of a metric Anosov flow. Again, the key issue is establishing the regularity properties (1) and (2) of Theorem A. We achieve this using similar ideas to the proof of Theorem B, although there are some additional technicalities since we must find machinery to stand in for the Busemann functions.

Theorem C. *The geodesic flow for a projective Anosov representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$, where Γ is a hyperbolic group admits a strong Markov coding.*

This result is a key step in the paper [BCLS15]. In that work, this statement is justified by showing that the flow is metric Anosov [BCLS15, Proposition 5.1] and then referencing [Pol87] as saying that this implies the existence of strong Markov coding. This claim also appears in the papers [BCS17, BCLS18, PS17, Sam16] either explicitly or implicitly through the claim that results that are true for Anosov flows are true for metric Anosov flows via [Pol87]. However, as discussed, [Pol87, Theorem 1] only provides a Markov coding with no guarantee of regularity of the roof function or projection map beyond continuity. When the phase space of the geodesic flow of the representation is a manifold, for example in the important case of Hitchin representations, the required regularity can be observed easily from smoothness of the flow and by taking smooth discs for sections in the construction of the symbolic dynamics, as Bowen argued in the Axiom A case. However, at this level of generality, this argument is not available. If Γ is not the fundamental group of a closed negatively curved manifold, then the phase space of the flow need not be a manifold. This is allowed in [BCLS15] in order to cover interesting and natural settings including deformation spaces of convex cocompact hyperbolic manifolds.

Our work provides an argument for the strong Markov coding at this level of generality which is then applied extensively in [BCLS15]; in particular, see [BCLS15, §3.5] for many interesting applications of the strong Markov coding in their setting. We emphasize that the paper [BCLS15] demonstrates Hölder structure for a variety of maps that arise naturally in this setting. Given this, it is the expected result that the Markov coding can be improved to the strong Markov coding; nevertheless, we believe that additional argument of the kind presented here is required. A major application in [BCLS15] is for their proof of analyticity of Hausdorff dimension over deformation spaces of convex cocompact hyperbolic manifolds.

The existence of a strong Markov coding allows one to instantly apply the rich array of results on dynamical and statistical properties from the literature that are proved for the suspension flow, and known to be preserved by the projection π. We collect some of these results as they apply to our primary example of the geodesic flow on a compact, locally $\mathrm{CAT}(-\kappa)$ space.

Corollary D. *For the geodesic flow on a compact, locally $\mathrm{CAT}(-\kappa)$ space, there exists a unique equilibrium measure μ_φ for every Hölder potential function φ on the space of geodesics. We have the following properties.*

1. *μ_φ satisfies the Almost Sure Invariance Principle, the Law of the Iterated Logarithm, and the Central Limit Theorem;*
2. *The dynamical zeta function is analytic on the region of the complex plane with real part greater than h, where h is the entropy of the flow, and has a meromorphic extension to points with real part greater than $h - \epsilon$.*
3. *If the lengths of periodic orbits are not all integer multiples of a single constant then the system is Bernoulli with respect to μ_φ;*

(4) *If the lengths of periodic orbits are all integer multiples of a single constant and the space is geodesically complete, then μ_φ is the product of Lebesgue measure for an interval with a Gibbs measure for an irreducible shift of finite type; the measure in the base is thus Bernoulli if the shift is aperiodic, or Bernoulli times finite rotation otherwise.*

The equilibrium measure for $\varphi = 0$ is the measure of maximal entropy, which is known in this setting as the *Bowen-Margulis measure* μ_{BM}. We give references for how the properties apply in §6. While items (1), (2), and (3) are true for any topologically transitive system with a strong Markov coding, item (4) additionally uses a structure theorem of Ricks in [Ric17], which applies for geodesic flow on geodesically complete CAT(0) spaces. Finally, we note that in our previous work [CLT16], we used a different approach based on the specification property to show that there is a unique equilibrium measure μ_φ. However, those techniques do not give the strong consequences listed above.

The paper is structured as follows. In §2, we establish our definitions and preliminary lemmas. In §3, we establish the machinery required to build a strong Markov coding for a metric Anosov flow, and prove Theorem A. In §4, we study geometrically defined sections to the flow, completing the proof of Theorem B. In §5, we extend the construction to projective Anosov representations, proving Theorem C. In §6, we discuss applications of the strong Markov coding, proving Corollary D.

2. Preliminaries

2.1. **Flows and sections.** We consider a continuous flow $\{\phi_t\}$ on a compact metric space (X, d). For a set D, and interval I, we write

$$\phi_I D = \{\phi_t x : x \in D,\ t \in I\}.$$

For a set $D \subset X$, we write $\operatorname{Int} D$ for the interior of D *transverse to the flow*; that is,

$$\operatorname{Int} D = D \cap \bigcap_{\epsilon > 0} (\phi_{(-\epsilon,\epsilon)} D)^\circ$$

where Y° denotes the interior of Y with respect to the topology of X.

We say that a flow $\{\phi_t\}$ is *Hölder continuous* if the map from $X \times [0,1] \to X$ given by $(x,t) \to \phi_t(x)$ is Hölder continuous. It follows that every time$-t$ map is Hölder continuous, and the map $t \to \phi_t(x)$ is Hölder continuous for each $x \in X$.

Definition 2.1. For a continuous flow $\{\phi_t\}$ on a metric space (X, d), a *section* is a closed subset $D \subset Y$ and a $\xi > 0$ so that the map $(z,t) \mapsto \phi_t z$ is a homeomorphism between $D \times [-\xi, \xi] \to X$ and $\phi_{[-\xi,\xi]} D$.

For any section D, there is a well-defined projection map $\operatorname{Proj}_D : \phi_{[-\xi,\xi]} D \to D$ defined by $\operatorname{Proj}_D(\phi_t z) = z$. By definition, the domain of this map contains a nonempty open neighborhood of X. In [BW72], a set $D \subset X$ is defined to be a section if D is closed and there exists $\xi > 0$ so that $D \cap \phi_{[-\xi,\xi]} x = \{x\}$ for all $x \in D$. It is easily checked that this is equivalent to Definition 2.1, see [BW72, §5].

2.2. **Shifts of finite type and suspension flow.** Let $\mathcal{A}$ be any finite set. The full, two-sided shift on the alphabet $\mathcal{A}$ is the dynamical system (Σ, σ) where

$$\Sigma = \{\underline{x} : \mathbb{Z} \to \mathcal{A}\} \quad \text{and} \quad (\sigma \underline{x})_n = \underline{x}_{n+1}.$$

We equip Σ with the metric

$$d(\underline{x},\underline{y}) := \frac{1}{2^l} \quad \text{where } l = \min\{|n| : \underline{x}_n \neq \underline{y}_n\}.$$

A *subshift* Y of the full shift is any closed, σ-invariant subset of Σ, equipped with the dynamics induced by σ. We say that (Y,σ) is a symbolic system. Given a $\{0,1\}$-valued $d\times d$ transition matrix A, where d is the cardinality of $\mathcal{A}$, a (1-step) subshift of finite type is defined by

$$\Sigma_A = \{\underline{x} \in \Sigma : A_{x_n x_{n+1}} = 1 \text{ for all } n \in \mathbb{Z}\}.$$

This is the class of symbolic spaces that appears in this paper. We now recall the suspension flow construction.

Definition 2.2. Given a symbolic system (Y,σ) and a positive function $\rho : Y \to (0,\infty)$, we let

$$Y^\rho = \{(\underline{x},t) : \underline{x} \in Y, 0 \le t \le \rho(\underline{x})\}/((\underline{x},\rho(\underline{x})) \sim (\sigma\underline{x},0))$$

and we define the suspension flow locally by $\phi_s(\underline{x},t) = (\underline{x},t+s)$. This is the *suspension flow over* (Y,σ) *with roof function* ρ. We denote the flow $(Y^\rho,\{\phi_s\})$ by $\mathrm{Susp}(Y,\rho)$.

2.3. CAT(-1) **spaces and geodesic flow.** A CAT(-1) space (X,d_X) is a geodesic metric space with the following property: Given any geodesic triangle $\Delta(x,y,z)$ in X, construct a comparison triangle $\Delta(\bar{x},\bar{y},\bar{z})$ in $\mathbb{H}^2$ having the same side lengths. Any points $p,q \in \Delta(x,y,z)$, determine comparison points $\bar{p},\bar{q}$ on $\Delta(\bar{x},\bar{y},\bar{z})$ having the same distances from the endpoints of the sides on which they lie. X is CAT(-1) if for all such p,q, $d_X(p,q) \le d_{\mathbb{H}^2}(\bar{p},\bar{q})$. In other words, a space is CAT(-1) if its geodesic triangles are thinner than corresponding triangles in the model space of curvature -1. A space (X,d) is said to be *locally* CAT(-1) if every point has a CAT(-1) neighborhood. The universal cover of a locally CAT(-1) space is (globally) CAT(-1) (see, e.g. [BH99, Thm II.4.1]). A CAT$(-\kappa)$ space is defined analogously: its geodesic triangles are thinner than corresponding triangles in the model space of curvature $-\kappa$. A CAT$(-\kappa)$ space can be rescaled homothetically to a CAT(-1) space. Thus, it suffices to consider CAT(-1) spaces.

The *boundary at infinity* of a CAT(-1) space is the set of equivalence classes of geodesic rays, where two rays are equivalent if they remain a bounded distance apart. We denote this boundary by $\partial^\infty X$. It can be equipped with the cone topology (see, e.g., [BH99, Chapter II.8]).

Definition 2.3. In any metric space (X,d_X), the *space of geodesics* is

$$GX := \{c : \mathbb{R} \to X \text{ where } c \text{ is a local isometry}\}.$$

The *geodesic flow* on GX is given by

$$g_t c(s) = c(s+t).$$

For a CAT(-1) space, $G\tilde{X}$ can be identified with $[(\partial^\infty\tilde{X}\times\partial^\infty\tilde{X})\setminus\Delta]\times\mathbb{R}$, where Δ is the diagonal. We endow $G\tilde{X}$ – the space of geodesics in the universal cover – with the following metric:

$$d_{G\tilde{X}}(\tilde{c},\tilde{c}') := \int_{-\infty}^{\infty} d_{\tilde{X}}(\tilde{c}(s),\tilde{c}'(s))e^{-2|s|}ds.$$

The factor 2 in the exponent normalizes the metric so that $d_{G\tilde{X}}(\tilde{c}, g_s\tilde{c}) = s$. The topologies induced on $G\tilde{X}$ by this metric and on $[(\partial^\infty\tilde{X} \times \partial^\infty\tilde{X}) \smallsetminus \Delta] \times \mathbb{R}$ using the cone topology on $\partial^\infty\tilde{X}$ agree. We endow GX with the metric

$$d_{GX}(c, c') = \inf_{\tilde{c}, \tilde{c}'} d_{G\tilde{X}}(c, c')$$

where the infimum is taken over all lifts $\tilde{c}, \tilde{c}'$ of c and c'. Since the set of lifts is discrete, the infimum is always achieved.

2.4. **Geometric lemmas.** The following lemma has an elementary proof which can be found in [CLT16, Lemma 2.9].

Lemma 2.4. *There exists some $L > 0$ such that $d_X(c(0), c'(0)) \le L d_{GX}(c, c')$.*

The following lemma shows that the time-t map of the geodesic flow is Lipschitz.

Lemma 2.5. *Fix any $T > 0$. Then for any $t \in [0, T]$, and any pairs of geodesics $x, y \in GX$,*

$$d_{GX}(g_t x, g_t y) < e^{2T} d_{GX}(x, y).$$

Proof. By definition, for properly chosen lifts,

$$d_{GX}(x, y) = \int_{-\infty}^{\infty} d_{\tilde{X}}(\tilde{x}(s), \tilde{y}(s)) e^{-2|s|} ds.$$

As $g_t\tilde{x}$ and $g_t\tilde{y}$ are lifts of $g_t x$ and $g_t y$, we compute:

$$\begin{aligned} d_{GX}(g_t x, g_t y) &\le \int_{-\infty}^{\infty} d_{\tilde{X}}(\tilde{x}(s+t), \tilde{y}(s+t)) e^{-2|s|} ds \\ &= \int_{-\infty}^{\infty} d_{\tilde{X}}(\tilde{x}(s), \tilde{y}(s)) e^{-2|s-t|} ds \\ &= \int_{-\infty}^{\infty} d_{\tilde{X}}(\tilde{x}(s), \tilde{y}(s)) e^{-2|s|} \cdot \frac{e^{-2|s-t|}}{e^{-2|s|}} ds \end{aligned}$$

It is easy to check that $\frac{e^{-2|s-t|}}{e^{-2|s|}} \le e^{2t}$, which completes the proof. □

It follows that the flow $\{g_t\}$ is Lipschitz, using Lemma 2.5 and the fact that $d_{GX}(g_s x, g_t x) = |s - t|$ for all x, and all s, t with $|s - t|$ sufficiently small.

2.5. **Busemann functions and horospheres.** We recall the definitions of Busemann functions and horospheres.

Definition 2.6. Let $\tilde{X}$ be a CAT(-1) space, $p \in \tilde{X}$ and $\xi \in \partial^\infty\tilde{X}$, and c the geodesic ray from p to ξ. The *Busemann function centered at ξ with basepoint p* is defined as

$$\begin{gathered} B_p(-, \xi) : \tilde{X} \to \mathbb{R}, \\ q \mapsto \lim_{t\to\infty} d_{\tilde{X}}(q, c(t)) - t. \end{gathered}$$

It is often convenient for us to use the geodesic ray $c(t)$ itself to specify the Busemann function centered at $c(+\infty)$ with basepoint $c(0)$. Thus, for a given geodesic ray $c(t)$, we say the *Busemann function determined by c* is the function

$$B_c(-) := B_{c(0)}(-, c(+\infty)).$$

It is an easy exercise to verify that any Busemann function is 1-Lipschitz, and it is a well-known fact that Busemann functions on CAT(-1) spaces are convex in the sense that for any geodesic η, $B_p(\eta(t), \xi)$ is a convex function of t (see, e.g. [BH99, Prop II.8.22]). The level sets for $B_p(-, \xi)$ are called *horospheres*.

2.6. **Stable and unstable sets for** $\mathrm{CAT}(-1)$ **spaces.** In a $\mathrm{CAT}(-1)$ space, we define strong stable and unstable sets in GX generalizing the strong stable and unstable manifolds for negatively curved manifolds.

Definition 2.7. Given $c \in GX$ with lift $\tilde{c} \in G\tilde{X}$, the *strong stable set through* c is

$$W^{ss}(c) = \mathrm{Proj}_{GX}\{c' \in G\tilde{X} : \tilde{c}'(\infty) = \tilde{c}(\infty) \text{ and } B_{\tilde{c}}(\tilde{c}'(0)) = 0\}.$$

For any $\delta > 0$, $W^{ss}_\delta(c)$, is $\{c' \in W^{ss}(c) : d_{GX}(c, c') < \delta\}$.

The *strong unstable set through* c is

$$W^{uu}(c) = \mathrm{Proj}_{GX}\{\tilde{c}' \in GX : \tilde{c}'(-\infty) = \tilde{c}(-\infty) \text{ and } B_{-\tilde{c}}(\tilde{c}'(0)) = 0\}$$

and $W^{uu}_\delta(c)$, is $\{c' \in W^{uu}(c) : d_{GX}(c, c') < \delta\}$, where $-\tilde{c}(t) = \tilde{c}(-t)$.

Lemma 2.8. *There exists a constant $C > 1$ so that for sufficiently small δ,*

(1) *if $c' \in W^{ss}_\delta(c)$ and $t > 0$, $d_{GX}(g_t c, g_t c') \leq C d_{GX}(c, c')e^{-t}$;*
(2) *if $c'' \in W^{uu}_\delta(c)$ and $t < 0$, $d_{GX}(g_t c, g_t c'') \leq C d_{GX}(c, c'')e^{t}$.*

Proof. We prove this for the stable sets in $\tilde{X}$. The result in X follows, and the proof for the unstable sets is analogous. First we note that in $\mathbb{H}^2$, given $\delta_0 > 0$, there exists $K > 1$ so that if $\bar{c}, \bar{c}'$ are two geodesics with $d_{\mathbb{H}^2}(\bar{c}(t), \bar{c}'(t)) < \delta_0$, $\bar{c}(\infty) = \bar{c}'(\infty) = \bar{\xi}$ and $\bar{c}(0), \bar{c}'(0)$ on the same horosphere centered at $\bar{\xi}$, then $d_{\mathbb{H}^2}(\bar{c}(t), \bar{c}'(t)) \leq K e^{-t} d_{\mathbb{H}^2}(\bar{c}(0), \bar{c}'(0))$ for all $t > 0$.

Let $c' \in W^{ss}_\delta(c)$ with δ small enough so that, via Lemma 2.4, $d_X(c(0), c'(0)) < \delta_0$. In $\tilde{X}$, consider the ideal triangle Δ with vertices $\tilde{c}(0)$, $\tilde{c}'(0)$ and $\tilde{c}(\infty) = \tilde{c}'(\infty) = \xi$. There exists an ideal comparison triangle $\bar{\Delta} = \Delta(\bar{c}(0), \bar{c}'(0), \bar{\xi})$ in $\mathbb{H}^2$ satisfying the $\mathrm{CAT}(-1)$ comparison estimate $d_{\tilde{X}}(\tilde{c}(t), \tilde{c}'(t)) \leq d_{\mathbb{H}^2}(\bar{c}(t), \bar{c}'(t))$, see [DSU17, Prop. 4.4.13]. We obtain for all $t \geq 0$,

$$d_{\tilde{X}}(\tilde{c}(t), \tilde{c}'(t)) \leq d_{\mathbb{H}^2}(\bar{c}(t), \bar{c}'(t)) \leq K d_{\tilde{X}}(\tilde{c}(0), \tilde{c}'(0))e^{-t}. \tag{2.1}$$

Now we calculate:

$$\begin{aligned} d_{GX}(g_t c, g_t c') \leq e^{-2t} &\int_{-\infty}^{0} d_{\tilde{X}}(\tilde{c}(u), \tilde{c}'(u))e^{-2|u|}du \\ &+ K e^{-t} \int_{-t}^{\infty} d_{\tilde{X}}(\tilde{c}(s), \tilde{c}'(s))e^{-2|s|}ds, \end{aligned}$$

by breaking our calculation of $d_{GX}(g_t c, g_t c')$ into integrals over $(-\infty, 0)$ and $(0, \infty)$, applying a change of variables to the first integral, and equation (2.1) to the second. We then have that $d_{GX}(g_t c, g_t c') \leq (1 + K)e^{-t} d_{GX}(c, c')$. □

3. Metric Anosov flows

In this section, we define metric Anosov flows and prove Theorem A.

3.1. **Metric Anosov flows.** A continuous flow on a compact metric space (Y, d) is a *metric Anosov flow*, also known as a *Smale flow*, if it is equipped with a topological notion of local product structure. That is, a bracket operation so that the point $< x, y >$ is analogous in the uniformly hyperbolic setting to the intersection of the unstable manifold of x with the strong stable manifold of y. We give the definition. For $\epsilon > 0$, let us write

$$(Y \times Y)_\epsilon := \{(x, y) \in Y \times Y : d(x, y) < \epsilon\}.$$

Assume there exists a constant $\epsilon > 0$ and a continuous map

$$< \,, \, > : (Y \times Y)_\epsilon \to Y,$$

which satisfies:

a) $< x, x >= x$
b) $<< x, y >, z >=< x, z >$
c) $< x, < y, z >>=< x, z >$.

We define the *local strong stable set according to* $< \, , \, >$ to be

$$V_\delta^{ss}(x) = \{u \mid u =< u, x > \text{ and } d(x, u) < \delta\},$$

and the *local unstable set according to* $< \, , \, >$ to be

$$V_\delta^{u}(x) = \{v \mid v =< x, v > \text{ and } d(x, v) < \delta\}.$$

We now define the *metric* strong stable and strong unstable sets as follows:

$$W_\delta^{ss}(x; C, \lambda) = \{v \in V_\delta^{ss}(x) \mid d(\phi_t x, \phi_t y) \le Ce^{-\lambda t} d(x, y) \text{ for } t \ge 0\}$$
$$W_\delta^{uu}(x; C, \lambda) = \{v \in V_\delta^{u}(x) \mid d(\phi_{-t} x, \phi_{-t} y) \le Ce^{-\lambda t} d(x, y) \text{ for } t \ge 0\}.$$

Definition 3.1. A flow $\{\phi_t\}$ is a *metric Anosov flow* if there exists $C, \lambda > 0$ and a continuous function $v : (Y \times Y)_\epsilon \to \mathbb{R}$, and a sufficiently small $\delta > 0$ such that $< \, , \, >: V_\delta^{ss}(x) \times V_\delta^{u}(x) \to X$ is a homeomorphism onto an open set in Y, and writing $W_\delta^{ss}(x) = W_\delta^{ss}(x; C, \lambda)$ and $W_\delta^{uu}(x) = W_\delta^{uu}(x; C, \lambda)$, we have

$$W_\delta^{uu}(\phi_{v(x,y)} x) \cap W_\delta^{ss}(y) = < x, y > .$$

Furthermore, $v(x, y)$ is the unique value of t so that $W_\delta^{uu}(\phi_t x) \cap W_\delta^{ss}(y)$ is non-empty.

We note that $< x, x >= x$ implies $v(x, x) = 0$ and then the continuity of $< \, , \, >$ and v imply that for any $\epsilon' \in (0, \epsilon)$, there exists $\delta > 0$ so that if $x, y \in (Y \times Y)_{\epsilon'}$, then $v(x, y) < \delta$ and

$$W_\delta^{uu}(\phi_{v(x,y)} x) \cap W_\delta^{ss}(y) = < x, y > .$$

The following property of metric Anosov flows follows the standard proof that Axiom A flows are expansive.

Theorem 3.2. ([Bow73, Cor 1.6], [Pol87, Prop 1]) *A metric Anosov flow satisfies the expansivity property.*

This is a corollary of the following result, which says that orbits that are close are exponentially close.

Theorem 3.3. ([Bow72, Proposition 1.6]) *For a metric Anosov flow, there are constants $C, \lambda > 0$ so that for all $\epsilon > 0$, there exists $\delta > 0$ so that if $x, y \in Y$ and $h : \mathbb{R} \to \mathbb{R}$ continuous such that $h(0) = 0$ and $d(\phi_t x, \phi_{h(t)} y) < \delta$ for all $t \in [-T, T]$, then $d(x, \phi_v y) < Ce^{-\lambda T}\delta$ for some $|v| < \epsilon$.*

Bowen's proof goes through without change in the setting of metric Anosov flows. In the case of geodesic flow on a CAT(-1) space, this is a well known property of geodesics in negative curvature: it holds for geodesics in $\mathbb{H}^2$ by standard facts from hyperbolic geometry, and this can be propagated to the universal cover of a locally CAT(-1) space by using two nearby geodesics to form a comparison quadrilateral in $\mathbb{H}^2$. The details of the argument in this case are contained in the proof of Proposition 4.3 of [CLT16].

Theorem 3.4. *For a compact locally* CAT(-1) *space X, the geodesic flow on $Y = GX$ is a metric Anosov flow.*

Proof. Let $\epsilon > 0$ be small enough that every $L\epsilon$-ball in X is (globally) CAT(-1), where L is given by Lemma 2.4. Then for any $(c, c') \in (GX \times GX)_\epsilon$, $c(0)$ and $c'(0)$ lie in a CAT(-1) open set U which lifts isometrically to $\tilde{U} \subset \tilde{X}$. Let $\tilde{c}$ and $\tilde{c}'$ be the lifts of these geodesics with basepoints in $\tilde{U}$. We define $< c, c' >$ to be the projection to GX of the geodesic $\tilde{d}$ with $\tilde{d}(+\infty) = \tilde{c}'(+\infty)$, $\tilde{d}(-\infty) = \tilde{c}(-\infty)$ and $B_{\tilde{c}'}(\tilde{d}(0)) = 0$ (see Figure 1).

It is easy to verify that $< \, , \, >$ is continuous and satisfies conditions (a), (b), and (c) from §3.1. $V^{ss}_\delta(c)$ consists of geodesics δ-close to c whose lifts with basepoint in $\tilde{U}$ have with the same forward endpoint as $\tilde{c}$ and have basepoint on $B_{\tilde{c}} = 0$, and $V^u_\delta(c)$ is geodesics δ-close to c whose lifts have the same backward endpoint as c. Using $G\tilde{X} \cong [(\partial^\infty \tilde{X} \times \partial^\infty \tilde{X}) \smallsetminus \Delta] \times \mathbb{R}$, it is clear that for sufficiently small δ, $< \, , \, >: V^s_\delta(c) \times V^u_\delta(c) \to GX$ is a homeomorphism onto an open neighborhood of c.

It follows from Lemma 2.8 that for a sufficiently large choice of C and $\lambda = 1$,

$$W^{ss}_\delta(c; C, \lambda) = \{c' : \tilde{c}'(+\infty) = \tilde{c}(+\infty),\ B_{\tilde{c}}(c'(0)) = 0,\ \text{and } d_{GX}(c, c') < \delta\};$$

$$W^{uu}_\delta(c; C, \lambda) = \{c' : \tilde{c}'(-\infty) = \tilde{c}(-\infty),\ B_{-\tilde{c}}(c'(0)) = 0,\ \text{and } d_{GX}(c, c') < \delta\}.$$

We then define $v : (GX \times GX)_\epsilon \to \mathbb{R}$ by setting $v(c, c')$ to be the negative of the signed distance along the geodesic $d =< c, c' >$ from its basepoint to the horocycle $B_{-c} = 0$. This is clearly continuous, and it is easily checked that

$$W^{uu}_\delta(g_{v(c,c')}c) \cap W^{ss}_\delta(c') =< c, c' >$$

and that for all other values of t, $W^{uu}_\delta(g_t c) \cap W^{ss}_\delta(c') = \varnothing$. $\square$

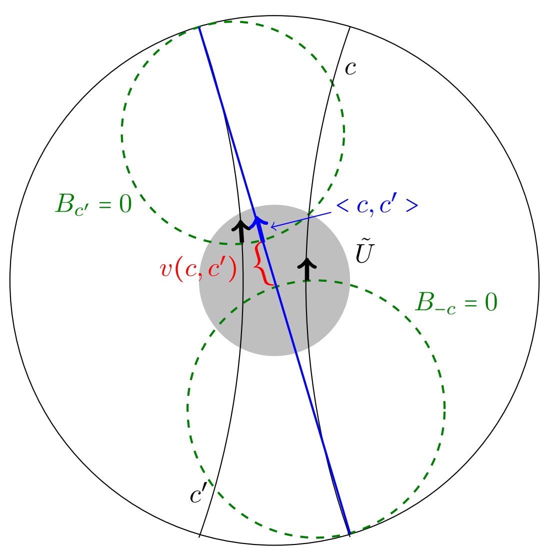

FIGURE 1. The geometric construction showing that geodesic flow on a CAT(-1) space is a metric Anosov flow.

3.2. **Sections, proper families, and symbolic dynamics for metric Anosov flows.** We recall the construction of a Markov coding for a metric Anosov flow. We follow the approach originally due to Bowen [Bow73] for Axiom A flows, which was shown to apply to metric Anosov flows by Pollicott [Pol87]. We recall Bowen's notion of a proper family of sections and a Markov proper family from [Bow73].

Definition 3.5. Let $\mathcal{B} = \{B_1, \dots, B_n\}$, and $\mathcal{D} = \{D_1, \dots, D_n\}$ be collections of sections. We say that $(\mathcal{B}, \mathcal{D})$ is a *proper family at scale* $\alpha > 0$ if $\{(B_i, D_i) : i = 1, 2, \dots, n\}$ satisfies the following properties:

(1) $\operatorname{diam}(D_i) < \alpha$ and $B_i \subset D_i$ for each $i \in \{1, 2, \dots, n\}$;
(2) $\bigcup_{i=1}^n \phi_{(-\alpha,0)}(\operatorname{Int} B_i) = Y$;
(3) For all $i \neq j$, if $\phi_{[0,4\alpha]}(D_i) \cap D_j \neq \varnothing$, then $\phi_{[-4\alpha,0]}(D_i) \cap D_j = \varnothing$.

Condition (3) implies that the sets D_i are pairwise disjoint, and the condition is symmetric under reversal of time; that is, it follows that if $\phi_{[-4\alpha,0]}(D_i) \cap D_j \neq \varnothing$, then $\phi_{[0,4\alpha]}(D_i) \cap D_j = \varnothing$. In [Bow73, Pol87], the time interval in condition (2) is taken to be $[-\alpha, 0]$. Our 'open' version of this condition is slightly stronger and convenient for our proofs in §4.2. We now define a special class of proper families, which we call *pre-Markov*.

Definition 3.6. For a metric Anosov flow, a *rectangle* R in a section D is a subset $R \subseteq \operatorname{Int} D$ such that for all $x, y \in R$, $\operatorname{Proj}_D < x, y >\in R$.

Definition 3.7 (Compare with §2 in [Pol87], §7 in [Bow73])**.** Let $(\mathcal{B}, \mathcal{D})$ be a proper family at scale $\alpha > 0$. We say that $(\mathcal{B}, \mathcal{D})$ is *pre-Markov* if the sets B_i are closed rectangles and we have the following following property:

$$\text{If } B_i \cap \phi_{[-2\alpha,2\alpha]} B_j \neq \varnothing, \text{ then } B_i \subset \phi_{[-3\alpha,3\alpha]} D_j. \tag{3.1}$$

The existence of pre-Markov proper families is left as an exercise by both Bowen and Pollicott since it is fairly clear that the conditions asked for are mild; some rigorous details are provided in [BW72]. In Proposition 4.10, we complete this exercise by providing a detailed proof of the existence of a special class of pre-Markov proper families. For our purposes, we must carry out this argument carefully since it is crucial for obtaining the Hölder return time property of Theorem A.

In [Pol87, Bow73], the following data is also added to a pre-Markov family: Let $\mathcal{K}$ be a collection of closed rectangles $K_i \subset \operatorname{Int} B_i$, and let $\delta > 0$ be chosen so any closed ball $\overline{B}(x, 6\delta)$ is contained in some $\phi_{[-2\alpha,2\alpha]} K_i$. Given a pre-Markov proper family, such a collection $\mathcal{K}$ and such a $\delta > 0$ can always be found. We can write $(\mathcal{K}, \mathcal{B}, \mathcal{D}, \delta)$ when the pre-Markov proper family $(\mathcal{B}, \mathcal{D})$ is equipped with this additional data.

We now define a Markov proper family. This is a proper family where the sections are rectangles, and with a property which can be informally stated as 'different forward $\mathcal{R}$-transition implies different future, and different backwards $\mathcal{R}$-transitions implies different past.'

Definition 3.8. A proper family $(\mathcal{R}, \mathcal{S})$ is *Markov* if the sets R_i are rectangles, and we have the following Markov property: let H denote the Poincaré return map for $\bigcup_i R_i$ with respect to the flow $\{\phi_t\}$. Then if $x \in R_i$ and $H(x) \in R_j$, and $z \in R_i$ and $H(z) \notin R_j$, then $z \notin V^{ss}_{\operatorname{diam} R_i}(x)$. Similarly, if $x \in R_i$ and $H^{-1}(x) \in R_j$, and $z \in R_i$ and $H^{-1}(z) \notin R_j$, then $z \notin V^{u}_{\operatorname{diam} R_i}(x)$.

The reason we call the families defined in Definition 3.7 *pre-Markov* is because the argument of §7 of [Bow73], and §2 of [Pol87] gives a construction to build Markov families out of pre-Markov families. The motivation for setting things up this way is that the existence of pre-Markov families can be seen to be unproblematic, whereas the existence of proper families with the Markov property is certainly non-trivial. More formally, we have:

Lemma 3.9. *If $(\mathcal{B}, \mathcal{D})$ is a pre-Markov proper family at scale α for a metric Anosov flow, then there exists a Markov proper family $(\mathcal{R}, \mathcal{S})$ so that for all i, there exists an integer j and a time u_j with $|u_i| << \alpha$ such that $R_i \subset \phi_{u_i} B_j$. The Markov proper family $(\mathcal{R}, \mathcal{S})$ can be constructed at an arbitrarily small scale $\alpha > 0$.*

Proof. This is the content of [Bow73, §7] in the case of Axiom A flows, and [Pol87, §2.2 'Key Lemma'] for metric Anosov flows. The proof involves cutting up sections from the pre-Markov family into smaller pieces; this can be carried out so that the resulting sections all have diameter less than α. The flow times u_i are used to push rectangles along the flow direction a small amount to ensure disjointness. These times can be taken arbitrarily small, in particular, much smaller than α. □

Note that if $\mathcal{B} = \{B_1, \dots, B_n\}$, then the collection $\mathcal{R} = \{R_1, \dots R_N\}$ provided by Lemma 3.9 satisfies $N >> n$.

3.3. **Markov partitions.** Given a collection of sections $\mathcal{R}$, let $H : \bigcup_{i=1}^N R_i \to \bigcup_{i=1}^N R_i$ be the Poincaré (return) map, and let $r : \bigcup_{i=1}^N R_i \to (0, \infty)$ be the return time function, which are well defined in our setting.

Definition 3.10. For a Markov proper family $(\mathcal{R}, \mathcal{S})$ for a metric Anosov flow, we define the coding space to be

$$\Sigma = \Sigma(\mathcal{R}) = \left\{ \underline{x} \in \prod_{-\infty}^{\infty} \{1, 2, \dots, N\} \mid \text{ for all } l, k \geq 0, \bigcap_{j=-k}^{l} H^{-j}(\operatorname{Int} R_{x_j}) \neq \varnothing \right\}.$$

In §2.3 of [Pol87], the symbolic space $\Sigma(\mathcal{R})$ is shown to be a shift of finite type. There is a canonically defined map $\pi : \Sigma(\mathcal{R}) \to \bigcup_i R_i$ given by $\pi(\underline{x}) = \bigcap_{j=-\infty}^{\infty} H^{-j}(R_{x_j})$. Let $\rho = r \circ \pi : \Sigma \to (0, \infty)$ and let $\Sigma^\rho = \Sigma^\rho(\mathcal{R})$ be the suspension flow over Σ with roof function ρ. We extend π to Σ^ρ by $\hat{\pi}(\underline{x}, t) = \phi_t(\pi(\underline{x}))$. Pollicott shows the following.

Theorem 3.11. ([Pol87, Theorem 1]) *If $\{\phi_t\}$ is a metric Anosov flow on Y, and $(\mathcal{R}, \mathcal{S})$ is a Markov proper family, then $\Sigma(\mathcal{R})$ is a shift of finite type and the map $\hat{\pi} : \Sigma^\rho \to Y$ is finite-to-one, continuous, surjective, injective on a residual set, and satisfies $\hat{\pi} \circ f_t = \phi_t \circ \hat{\pi}$, where $\{f_t\}$ is the suspension flow.*

We say that a flow has a *strong Markov coding* if the conclusions of the previous theorem are true with the additional hypothesis that the roof function ρ is Hölder and that the map $\hat{\pi}$ is Hölder. This is condition (III) on p.195 of [Pol87]. Since $\rho = r \circ \pi : \Sigma \to (0, \infty)$, it suffices to know that $\hat{\pi}$ is Hölder and r is Hölder where it is continuous. Thus, we can formulate Pollicott's result as follows:

Theorem 3.12 (Pollicott). *If $\{\phi_t\}$ is a metric Anosov flow, and there exists a Markov proper family $(\mathcal{R}, \mathcal{D})$ such that the return time function r for $\mathcal{R}$ is Hölder where it is continuous, and the natural projection map $\hat{\pi} : \Sigma^\rho \to X$ is Hölder, then the flow has a strong Markov coding.*

A drawback of this statement is that it is not clear how to meet the Hölder requirement of these hypotheses. Our Theorem A is designed to remedy this. Recall the hypotheses of Theorem A are that the metric Anosov flow is Hölder and that there exists a pre-Markov proper family $(\mathcal{B}, \mathcal{D})$ so that the return time function and the projection maps to the B_i are Hölder. We now prove Theorem A by showing that these hypotheses imply the hypotheses of Theorem 3.12.

Proof of Theorem A. We verify the hypotheses of Theorem 3.12. Let the family $(\mathcal{R}, \mathcal{S})$ be the Markov family provided by applying Lemma 3.9 to $(\mathcal{B}, \mathcal{D})$. Recall that by Lemma 3.9, we can choose the scale α for $(\mathcal{B}, \mathcal{D})$ as small as we like. Then $\mathcal{R}$ consists of rectangles R_i which are subsets of elements of $\mathcal{B}$ shifted by the flow for some small time. Thus, the return time function for $\mathcal{R}$ inherits Hölder regularity from the return time function for $\mathcal{B}$.

Now we use Theorem 3.3 to show that the projection map π from $\Sigma(\mathcal{R})$ is Hölder. Fix some small $\alpha_0 > 0$. Choose $\epsilon > 0$ sufficiently small that the projection maps to any section S with diameter $< \alpha_0$ are well-defined on $\phi_{[-\epsilon,\epsilon]}S$. Then let us suppose that our Markov family is at scale α so small that $\alpha < \alpha_0$ and $3\alpha < \delta$ where δ is given by Theorem 3.3 for the choice of ϵ above. Let $\underline{i}, \underline{j} \in \Sigma(\mathcal{R})$ which agree from i_{-n} to i_n. We write x, y for the projected points, which belong to some B_{i^*}. If two orbits pass through an identical finite sequence $R_{i_{-n}}, \ldots, R_{i_0}, \ldots R_{i_n}$ then they are 3α-close for time at least $2n$ multiplied by the minimum value of the return map on $\mathcal{R}$, which we write r_0. The distance is at most 3α since $\operatorname{diam} R_i < \alpha$ and the return time is less than α. Thus, by Theorem 3.3 there is a time v with $|v| < \epsilon$ so that $d(x, \phi_v y) < \alpha e^{-\lambda 2 n r_0}$. Using Hölder continuity of the projection map $\mathrm{Proj}_{R_{i_0}}$, which is well-defined at $\phi_v y$ since $|v| < \epsilon$, we have

$$d(x, y) = d(\mathrm{Proj}_{R_{i_0}} x, \mathrm{Proj}_{R_{i_0}} \phi_v y) < C d(x, \phi_v y)^{\beta},$$

where β is the Hölder exponent for the projection map. Thus, $d(x, y) < C\alpha e^{-(2\beta\lambda r_0)n}$. Since $d(\underline{i}, \underline{j}) = 2^{-n}$, this shows the projection π from $\Sigma(\mathcal{R})$ is Hölder.

It follows that the roof function $\rho = \pi \circ r$ is Hölder. Thus, since π is Hölder, the roof is Hölder and the flow is Hölder, it follows that $\hat{\pi} : \Sigma^{\rho} \to X$ is Hölder. □

The advantage of the formulation of Theorem A is that the hypotheses for the strong Markov coding are now written entirely in terms of properties of the flow and families of sections $\mathcal{D}$. In the terminology introduced above, Bowen showed that transitive Axiom A flows admit a strong Markov coding, using smoothness of the flow and taking the sections to be smooth discs to obtain the regularity of the projection and return maps. For a Hölder continuous metric Anosov flow, we do not know of a general argument to obtain this regularity. Our strategy to verify the hypotheses of Theorem A in the case of geodesic flow on a CAT(-1) space is to construct proper families in which the sections are defined geometrically. For these special sections, we can establish the regularity that we need. Our argument relies heavily on geometric arguments which are available for CAT(-1) geodesic flow, but do not apply to general metric Anosov flows.

4. Geometric rectangles and Hölder properties

In this section, we define geometric rectangles which can be built in $G\tilde{X}$ for any CAT(-1) space $\tilde{X}$.

Definition 4.1. Let U^+ and U^- be disjoint open sets in $\partial_\infty \tilde{X}$. Let $T \subset \tilde{X}$ be a transversal on $\tilde{X}$ to the geodesics between U^- and U^+ – that is, a set T so any geodesic c with $c(\infty) \in U^+$ and $c(-\infty) \in U^-$ intersects T exactly once. Let $R(T, U^+, U^-)$ be the set of all geodesics c with $c(\infty) \in U^+$ and $c(-\infty) \in U^-$ and which are parametrized so that $c(0) \in T$. If $R(T, U^+, U^-)$ is a section to the geodesic flow, we call $R(T, U^+, U^-)$ a *geometric rectangle*.

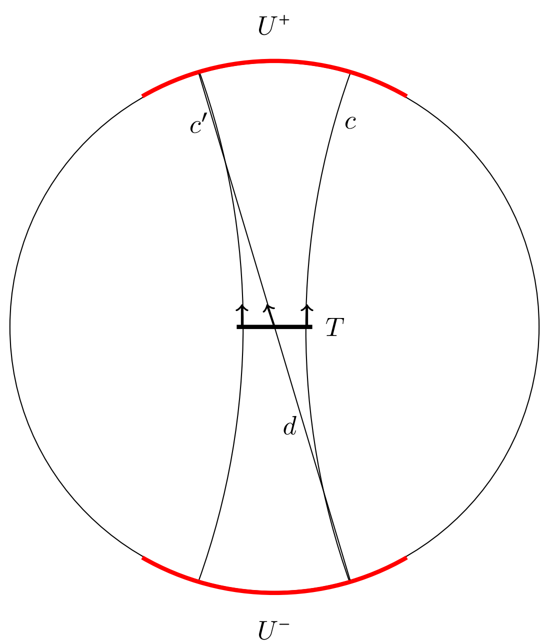

FIGURE 2. Illustrating Definition 4.1. The arrows mark the base-point and direction for each geodesic in $R(T, U^+, U^-)$.

If $c, c' \in R(T, U^+, U^-)$, then $\mathrm{Proj}_T < c, c' >$ is the geodesic d which connects the backwards endpoint of c to the forwards endpoint of c', with $d(0) \in T$, and thus $R(T, U^+, U^-)$ is a rectangle in the sense of Definition 3.6. See Figure 2.

To build rectangles we need to specify the sets U^+ and U^- and choose our transversals. We do so in the following definition.

Fix a parameter $\tau >> 1$. Let $c \in G\tilde{X}$. Let $B_1 = B_{d_{\tilde{X}}}(c(-\tau), 1)$ and $B_2 = B_{d_{\tilde{X}}}(c(\tau), 1)$ be the open balls of $d_{\tilde{X}}$-radius 1 around $c(\pm\tau)$. Let

$$\gamma(c, \tau) = \{c' \in GX : c' \cap B_i \neq \varnothing \text{ for } i = 1, 2\}.$$

Let

$$\partial(c, \tau) = \{(c'(-\infty), c'(+\infty)) \in \partial^\infty \tilde{X} \times \partial^\infty \tilde{X} : c' \in \gamma(c, \tau)\}.$$

It is easy to check that $\partial(c, \tau)$ is open in the product topology on $\partial^\infty \tilde{X} \times \partial^\infty \tilde{X}$. Then we may find open sets U^- and U^+ such that $(c(-\infty), c(+\infty)) \in U^- \times U^+ \subset \partial(c, \tau)$.

Definition 4.2. Let $c \in G\tilde{X}$ and $\tau >> 1$. Let U^- and U^+ satisfy $U^- \times U^+ \subset \partial(c, \tau)$. The *good rectangle* $R(c, \tau; U^-, U^+)$ is the set of all $\eta \in G\tilde{X}$ which satisfy:

(1) $\eta(-\infty) \in U^-$ and $\eta(+\infty) \in U^+$,
(2) $B_c(\eta(0)) = 0$,
(3) If $\eta(t_1) \in B_1$ and $\eta(t_2) \in B_2$, then $t_1 < 0 < t_2$.

To remove arbitrariness in the choice of U^-, U^+, we can let $\delta > 0$ be the biggest value so that if $U^-_\delta = B_\infty(c(+\infty), \delta)$ and $U^+_\delta = B_\infty(c(-\infty), \delta)$, then $U^-_\delta \times U^+_\delta \subset \partial(c, \tau)$. We can set $R(c, \tau) = R(c, \tau; U^-_\delta, U^+_\delta)$.

In other words, for *good* rectangles, we take as our transversal T on X a suitably sized disc in the horocycle based at $c(+\infty)$ through $c(0)$ (see Figure 3).

We will usually consider the 'maximal' good rectangle $R(c, \tau)$. However, we note that the definition makes sense for any $V^- \times V^+ \subset \partial(c, \tau)$. In particular, it is not required that the geodesic c itself (which defines the horocycle that specifies the parameterization of the geodesics) be contained in $R(c, \tau; V^-, V^+)$.

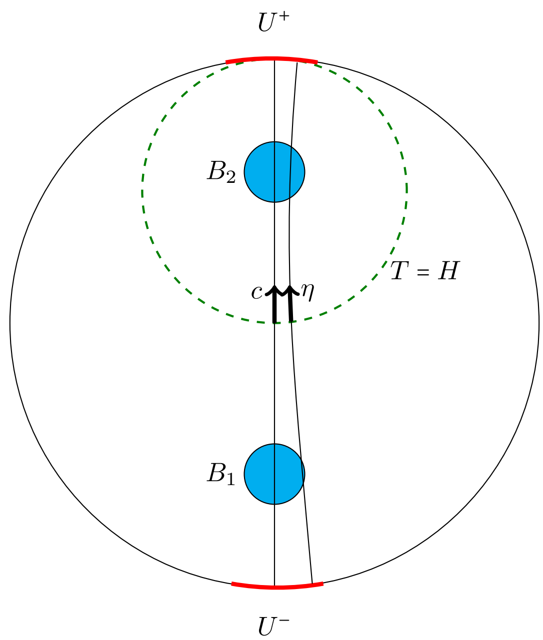

FIGURE 3. A geodesic $\eta \in R(c, \tau; U^-, U^+)$ as in Definition 4.2.

To justify this definition, we must verify that $R(c, \tau; U^-, U^+)$ is in fact a rectangle in the sense of Definition 4.1. That is, we need to prove the following two lemmas:

Lemma 4.3. *For any $\eta \in G\tilde{X}$ with $\eta(-\infty) \in U^-$ and $\eta(+\infty) \in U^+$, there is exactly one point $p \in \eta$ such that $B_c(p) = 0$ and such that p lies between η's intersections with B_1 and B_2.*

Proof. We have $B_{c(0)}(\eta(t_1), c(+\infty)) > 0$ when $\eta(t_1) \in B_1$ and $B_{c(0)}(\eta(t_2), c(+\infty)) < 0$ when $\eta(t_2) \in B_2$. Continuity and convexity of the Busemann function implies that there is a unique $t^* \in (t_1, t_2)$ such that $B_{c(0)}(\eta(t^*), c(+\infty)) = 0$. Let $p = \eta(t^*)$. ☐

Lemma 4.4. *$R(c, \tau; U^-, U^+)$ is a section.*

Proof. The openness of U^- and U^+, and the 1-Lipschitz property of Busemann functions are the key facts. ☐

We give the following distance estimates for geodesics in a rectangle.

Lemma 4.5. *For all $\eta \in R(c,\tau;U^-,U^+)$, we have*

(1) $d_{\tilde{X}}(c(0),\eta(0)) \le 2$;
(2) $d_{\tilde{X}}(c(\pm\tau),\eta(\pm\tau)) < 4$.

Proof. First, we prove (1). By the definition of the rectangle, we know that there exist times $t^+ > 0$ and $t^- < 0$ so that $d_{\tilde{X}}(c(\tau),\eta(t^+)) < 1$ and $d_{\tilde{X}}(c(-\tau),\eta(t^-)) < 1$. Since the distance between two geodesic segments is maximized at one of the endpoints, we know that $d_{\tilde{X}}(\eta(0),c) < 1$. Thus, there exists t^* so that $d_{\tilde{X}}(\eta(0),c(t^*)) < 1$. Thus, $d_{\tilde{X}}(c(0),\eta(0)) \le d_{\tilde{X}}(c(0),c(t^*)) + d_{\tilde{X}}(\eta(0),c(t^*)) < |t^*| + 1$.

Since the Busemann function is 1-Lipschitz,

$$|B_c(c(t^*))| = |B_c(c(t^*)) - B_c(\eta(0))| \le d_{\tilde{X}}(c(t^*),\eta(0)) < 1.$$

Since $|B_c(c(t^*))| = |t^*|$, it follows that $|t^*| < 1$. Thus, $d_{\tilde{X}}(c(0),\eta(0)) < 2$.

We use (1) to prove (2). Observe that $t^+ \le \tau + 3$. This is because

$$\begin{aligned} t^+ = d_{\tilde{X}}(\eta(0),\eta(t^+)) &\le d_{\tilde{X}}(\eta(0),c(0)) + d_{\tilde{X}}(c(0),c(\tau)) + d_{\tilde{X}}(c(\tau),\eta(t^+)) \\ &\le 2 + \tau + 1. \end{aligned}$$

We also see that $t^+ \ge \tau - 3$. This is because

$$\begin{aligned} \tau = d_{\tilde{X}}(c(0),c(\tau)) &\le d_{\tilde{X}}(c(0),\eta(0)) + d_{\tilde{X}}(\eta(0),\eta(t^+)) + d_{\tilde{X}}(\eta(t^+),c(\tau)) \\ &\le 2 + t^+ + 1. \end{aligned}$$

Thus $|\tau - t^+| < 3$. It follows that

$$d_{\tilde{X}}(c(\tau),\eta(\tau)) \le d_{\tilde{X}}(c(\tau),\eta(t^+)) + d_{\tilde{X}}(\eta(\tau),\eta(t^+)) < 1 + 3 = 4.$$

The argument that $d_{\tilde{X}}(c(-\tau),\eta(-\tau)) < 4$ is analogous. □

We obtain linear bounds on the Busemann function for $\eta \in R(c,\tau;U^-,U^+)$.

Lemma 4.6. *For all $\eta \in R(c,\tau;U^-,U^+)$,*

$$-t \le B_{c(0)}(\eta(t),c(+\infty)) \le -\frac{t}{2} \text{ for all } 0 \le t < \tau$$

and

$$-\frac{t}{2} \le B_{c(0)}(\eta(t),c(+\infty)) \le -t \text{ for all } -\tau < t \le 0.$$

That is, for times between $-\tau$ and τ, the values of the Busemann function along η lie between $-t$ and $-\frac{t}{2}$.

Proof. That $-t \le B_c(\eta(t))$ follows immediately from the 1-Lipschitz property of Busemann functions. By Lemma 4.5 and the 1-Lipschitz property of Busemann functions, $|B_c(\eta(\tau)) + \tau| = |B_c(\eta(\tau)) - B_c(c(\tau))| < 4$, and similarly $|B_c(\eta(\tau)) - \tau| = |B_c(\eta(-\tau)) - B_c(c(-\tau))| < 4$. Therefore, $f(t) = B_c(\eta(t))$ is a convex function with $f(-\tau) \in (\tau-4,\tau]$, $f(0) = 0$ and $f(\tau) \in [-\tau,-(\tau-4))$. Then if for some $t_0 \in (-\tau,0]$, $f(t_0) < -\frac{t_0}{2}$, or for some $t_0 \in [0,\tau)$, $f(t_0) > -\frac{t_0}{2}$, then for all $t > \max\{0,t_0\}$, by convexity, $f(t) > -t/2$. But then $f(\tau) > -\frac{\tau}{2}$, a contradiction since $\tau >> 1$. □

The proof actually yields the upper bound of $B_c(\eta(t)) \le -\frac{\tau-4}{\tau}t$ but all we need is some linear bound with non-zero slope.

Lemma 4.7. *Let R_1 and R_2 be rectangular subsets of good geometric rectangles. Suppose* $\operatorname{diam}(R_1) = \epsilon$ *and that $R_1 \cap R_2 \neq \varnothing$. Then for $|t| > 2L\epsilon$, $g_t R_1 \cap R_2 = \varnothing$, where L is the constant from Lemma 2.4.*

Proof. Let f be the Busemann function used to specify the basepoints of geodesics in R_2. Since the diameter of R_1 is ϵ, and since for some $\eta \in R_1$, $f(\eta(0)) = 0$, $|f(c(0))| < L\epsilon$ for all $c \in R_1$. This uses Lemma 2.4 and the 1-Lipschitz property of Busemann functions with respect to d_X. If $\eta \in R_1 \cap R_2$, then $\frac{1}{2}|t| \leq |f(\eta(t))| \leq |t|$ by Lemma 4.6. Now suppose that $\eta \in g_t R_1 \cap R_2$ for some $|t| > 2L\epsilon$. Then $g_{-t}\eta \in R_1$ and we must have $|f(\eta(-t))| > L\epsilon$, which is a contradiction. □

4.1. **Hölder properties.** We are now ready to prove the regularity results we need to apply Theorem A. First, we show that return times between geometric rectangles are Lipchitz. Let $R = R(c, \tau; U^+, U^-)$ and $R' = R(c', \tau'; U'^+, U'^-)$ be good geometric rectangles and let $d \in R$ such that $g_{t_0} d \in R'$ for some t_0 which is minimal with respect to this property. We write $r(d) = r(d, R, R') := t_0$; this is the return time for d to $R \cup R'$.

Let us make the standing assumption that all return times are bounded above by $\alpha > 0$. Note that $d \in R$ and $g_{t_0} d \in R'$ iff $d(-\infty) \in U^- \cap U'^-$ and $d(+\infty) \in U^+ \cap U'^+$. The key property we want is the following:

Proposition 4.8. *Let R, R' be good rectangles and $Y = R \cap H^{-1}(R')$. Then the return time map $r : (Y, d_{GX}) \to \mathbb{R}$ is Lipschitz.*

Proof. Let $v, w \in Y$ with return times $r(v), r(w)$, respectively. Let $\epsilon = d_{GX}(v, w)$. We consider the Busemann function determined by the geodesic c' which defines the rectangle R'.

Let $f(t) = B_{c'}(v(r(v) + t))$ and let $g(t) = B_{c'}(w(r(v) + t))$. Then $r(w) - r(v) = t^*$ where t^* is the unique value of t with $|t^*| < \alpha$ such that $g(t^*) = 0$. By Lemma 4.6, the graph of $f(t)$ lies between the lines $y = -t$ and $y = -\frac{t}{2}$ for small t.

Let $C = e^{\alpha}$, where α is an upper bound on the return time. By Lemmas 2.4 and 2.5, $d_X(v(s), w(s)) < LC\epsilon$ for all $s < \alpha$, where C is a uniform constant. The 1-Lipschitz property of Busemann functions implies that $|f(t) - g(t)| < LC\epsilon$.

Thus, for $t > 0$, we have $g(t) \leq f(t) + LC\epsilon \leq -t/2 + LC\epsilon$, and so for $t > 2LC\epsilon$, $g(t) < 0$. For $t < 0$, we have $g(t) \geq f(t) - LC\epsilon \geq -t/2 - LC\epsilon$, and so for $t < -2LC\epsilon$, we have $g(t) > 0$. Thus, by the intermediate value theorem, the root $g(t^*) = 0$ satisfies $t^* \in (-2LC\epsilon, 2LC\epsilon)$. Therefore, $|r(w) - r(v)| = |t^*| < 2LC\epsilon$ proving the desired Lipschitz property with constant $2LC$. □

We now show that the projection map to a good rectangle is Hölder. Consider any good geometric rectangle $R = R(c, \tau; U^-, U^+)$. Fix some small $\alpha > 0$ so that $(-\alpha, \alpha) \times R \to G\tilde{X}$ by $(t, x) \mapsto g_t x$ is a homeomorphism.

Proposition 4.9. $\mathrm{Proj}_R : g_{(-\alpha,\alpha)} R \to R$ *is Hölder.*

Proof. We prove that for all $x, y \in g_{(-\alpha,\alpha)} R$ there exists some $K > 0$ such that

$$d_{G\tilde{X}}(\mathrm{Proj}_R x, \mathrm{Proj}_R y) \leq K d_{G\tilde{X}}(x, y)^{\frac{1}{2}}.$$

First, note that for all $|t| < 2\alpha$, g_t is a $e^{2\alpha}$-Lipschitz map by Lemma 2.5. Therefore, to prove the Proposition, it suffices to prove the case where $x \in R$, as we can pre-compose the projection in this case with the Lipschitz map g_{t^*} where $g_{t^*} x \in R$.

Let $t = B_c(y(0))$. By Lemma 4.6 for all $|s| < \tau$, $\frac{|s|}{2} \leq |B_c(y(s)) - t| \leq |s|$. Similarly, $\frac{|s|}{2} \leq |B_c(x(s))| \leq |s|$. Since $B_c(x(s))$ and $B_c(y(s))$ are both decreasing by definition of R, these inequalities give us that

$$|B_c(x(s)) - B_c(y(s))| \geq t - \frac{|s|}{2} \quad \text{for all } s \leq \tau.$$

Since B_c is a 1-Lipschitz function on $\tilde{X}$,

$$d_{\tilde{X}}(x(s), y(s)) \geq t - \frac{|s|}{2} \quad \text{for all } s \leq \tau.$$

Then we can compute

$$d_{G\tilde{X}}(x,y) \geq \int_{-2t}^{2t} \left(t - \frac{|s|}{2}\right) e^{-2|s|} ds = \frac{1}{4}(-1 + e^{-4t} + 4t) \geq ct^2$$

for a properly chosen $c > 0$ since $|t| < \alpha$. By Lemma 4.6 and the fact that the geodesic flow moves at speed one for $d_{G\tilde{X}}$, $d_{G\tilde{X}}(y, \operatorname{Proj}_R y) \leq 2|t|$. Using $d_{G\tilde{X}}(x, \operatorname{Proj}_R y) \leq d_{G\tilde{X}}(x,y) + d_{G\tilde{X}}(y, \operatorname{Proj}_R y)$, if there exists some $L > 0$ such that $d_{G\tilde{X}}(y, \operatorname{Proj}_R y) \leq L d_{G\tilde{X}}(x,y)^{1/2}$, the Lemma is proved. But we have shown above that $d_{G\tilde{X}}(x,y) \geq ct^2$ and $d_{G\tilde{X}}(y, \operatorname{Proj}_R y) \leq 2|t|$. $\square$

4.2. **A pre-Markov proper family of good rectangles.** To complete our argument, it suffices to check that a pre-Markov proper family $(\mathcal{R}, \mathcal{S})$ can be found where the family of sections $\mathcal{S}$ consists of good geometric rectangles, perhaps flowed by a small time. Applying the results of the previous section, this will show that $(\mathcal{R}, \mathcal{S})$ has properties (1) and (2) of Theorem A.

Proposition 4.10. *For the geodesic flow $\{g_t\}$ on a locally $\mathrm{CAT}(-\kappa)$ space, for any sufficiently small $\alpha > 0$, there exists a pre-Markov proper family $(\mathcal{B}, \mathcal{D})$ at scale α such that each D_i has the form $g_{s_i} R_i$ for some s_i with $|s_i| << \alpha$ and some good geometric rectangle R_i.*

We need the following lemma.

Lemma 4.11. *Let $\{\phi_t\}$ be a Lipschitz continuous expansive flow on a compact metric space. Given a proper family $(\mathcal{B}, \mathcal{D})$ for $\{\phi_t\}$ at scale $\alpha > 0$ where the B_i and D_i are rectangles, there exists a pre-Markov proper family $(\mathcal{B}', \mathcal{D}')$ at scale $\alpha > 0$ such that every $D'_k \in \mathcal{D}'$ is the image under ϕ_{s_k} of some $D_i \in \mathcal{D}$ where $|s_k| << \alpha$.*

It is clear from the proof below that the times s_k can be made arbitrarily small in absolute value.

Proof. Let $(\mathcal{B}, \mathcal{D}) = \{(B_i, D_i) : i = 1, \ldots, n\}$ be a proper family at scale α where the B_i and D_i are rectangles. Recall that by definition, a proper family satisfies: (1) $\operatorname{diam}(D_i) < \alpha$ and $B_i \subset D_i$ for each $i \in \{1, 2, \ldots, n\}$; (2) $\bigcup_{i=1}^n \phi_{(-\alpha,0)}(\operatorname{Int} B_i) = Y$; (3) for all $i \neq j$, if $\phi_{[0,4\alpha]}(D_i) \cap D_j \neq \varnothing$, then $\phi_{[-4\alpha,0]}(D_i) \cap D_j = \varnothing$.

Our strategy for constructing new proper families out of $(\mathcal{B}, \mathcal{D})$ is to replace an element (B_i, D_i) by a finite collection $\{(\phi_{s_k} R_k, \phi_{s_k} D_i)\}_k$ where R_k are rectangles with $R_k \subset B_i$ and $\bigcup_k \operatorname{Int}(R_k)$ covers Int B_i. Then $\phi_{s_k} R_k$ and $\phi_{s_k} D_i$ inherit the rectangle property from R_k and D_i (and are closed if R_k and D_i are), and by choosing all s_k distinct and sufficiently small in absolute value, we can ensure that the resulting collection will still satisfy (1), (2), and (3). We give some details.

For (1), since the flow is Lipschitz and $\operatorname{diam}(D_i) < \alpha$, we can choose ϵ_1 so small that $\operatorname{diam}(\phi_{\pm\epsilon_1} D_i) < \alpha$. Thus, (1) will be satisfied if all s_k have $|s_k| < \epsilon_1$.

For (2), since $\bigcup_k \operatorname{Int}(R_k)$ covers Int B_i, then it suffices to assume that all s_k are sufficiently small in absolute value.

For (3), let $\beta > 0$ be the minimum value of s so that there is a pair D_j, D_k in our proper family with both $\phi_{[0,s]} D_j \cap D_k$ and $\phi_{[-s,0]} D_j \cap D_k$ nonempty, and observe that we must have $\beta > 4\alpha$. Choosing ϵ_2 smaller than $\frac{\beta - 4\alpha}{2}$ and smaller than

$d(D_j, D_k)$ for any $j \neq k$, condition (3) will be satisfied for $\phi_{s_j} D_j$ and $\phi_{s_k} D_k$ when $|s_j|, |s_k| < \epsilon_2$ and $j \neq k$. Also, it is clear that (3) will hold for the pair $\phi_{s_{k_1}} D$ and $\phi_{s_{k_2}} D$ when $D \in \mathcal{D}$, $|s_{k_1}|, |s_{k_2}| < \epsilon_2$ and $s_{k_1} \neq s_{k_2}$.

We now use this strategy to refine $(\mathcal{B}, \mathcal{D})$ to ensure the pre-Markov property (3.1) holds. For $B \in \mathcal{B}$, consider the set

$$F(B; \mathcal{B}, \mathcal{D}) = \{B_j \in \mathcal{B} :\ B \cap \phi_{[-2\alpha, 2\alpha]} B_j \neq \varnothing \text{ but } B \notin \phi_{(-3\alpha, 3\alpha)} D_j\}.$$

The set $F(B; \mathcal{B}, \mathcal{D})$ is finite and encodes the elements of the proper family for which an intersection with B causes an *open* version of (3.1) to fail. Clearly if $F(B; \mathcal{B}, \mathcal{D}) = \varnothing$ for all $B \in \mathcal{B}$, then the pre-Markov condition (3.1) is satisfied.

Let $i_1 < i_2 < \cdots < i_n$ be the set of all indices so that $F(B_{i_j}; \mathcal{B}, \mathcal{D}) \neq \varnothing$. We cover B_{i_1} by a finite collection of rectangles $R_k \subset B_{i_1}$ such that

- $\bigcup_k \operatorname{Int}(R_k)$ covers $\operatorname{Int} B_{i_1}$, and
- If $R_k \cap \phi_{[-2\alpha, 2\alpha]} B_j \neq \varnothing$ for some j, then $R_k \subseteq \phi_{(-3\alpha, 3\alpha)} D_j$.

It is clearly possible to find collections of rectangles satisfying the first condition. The second can be satisfied because $B_{i_1} \cap \phi_{[-2\alpha, 2\alpha]} B_j$ is a closed subset of B_{i_1} contained in the open subset $\phi_{(-3\alpha, 3\alpha)} D_j$, with respect to the subspace topology on B_{i_1}. We replace (B_{i_1}, D_{i_1}) in $(\mathcal{B}, \mathcal{D})$ with $\{(\phi_{s_k} R_k, \phi_{s_k} D_{i_1})\}_k$ for distinct times s_k sufficiently small in absolute value as detailed above. We obtain $(\mathcal{B}^1, \mathcal{D}^1)$ with $\mathcal{B}^1$ consisting of closed rectangles satisfying conditions (1), (2), and (3).

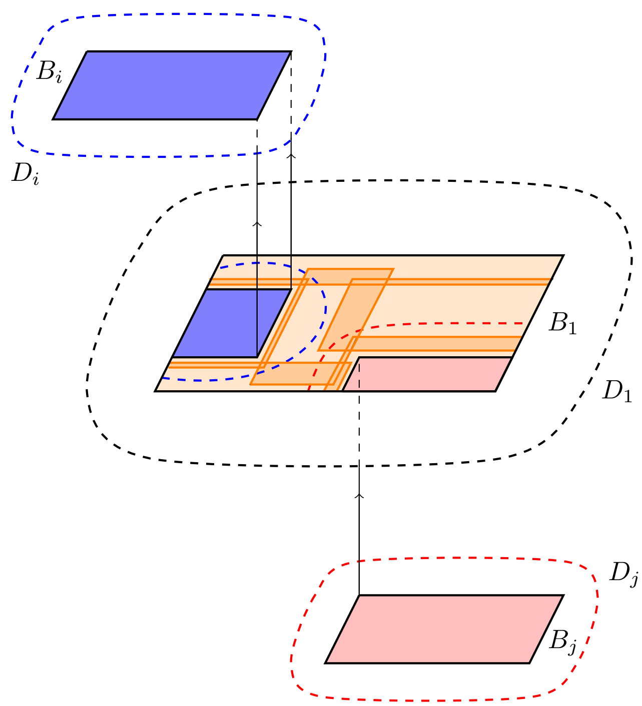

FIGURE 4. Ensuring the pre-Markov condition (3.1). $(1, j)$ and $(1, i)$ belong to $F(B_1; \mathcal{B}, \mathcal{D})$. The flow direction is vertical. The orange rectangles provide one possible choice for the R_k.

To establish condition (3.1), we must prove two things. First, we claim that for all k, $F(\phi_{s_k} R_k; \mathcal{B}^1, \mathcal{D}^1) = \varnothing$. This is true for the following reasons. First, if $B_j \in \mathcal{B}$ with $i_1 \neq j$, then by construction we know that $B_j \notin F(\phi_{s_k} R_k; \mathcal{B}^1, \mathcal{D}^1)$. It remains only to consider sets of the form $\phi_{s_i} R_i$ with $i \neq k$. Suppose $\phi_{s_k} R_k \cap \phi_{[-2\alpha,2\alpha]} \phi_{s_i} R_i \neq \varnothing$. Then it is clear that since $|s_i|, |s_k|$ are small and $R_k \subset D_{i_1}$, then $\phi_{s_k} R_k \subset \phi_{(-3\alpha,3\alpha)} \phi_{s_i} D_{i_1}$. It follows that $\phi_{s_i} R_i \notin F(\phi_{s_k} R_k; \mathcal{B}^1, \mathcal{D}^1)$. We conclude that $F(\phi_{s_k} R_k; \mathcal{B}^1, \mathcal{D}^1) = \varnothing$. That is, we have eliminated the 'bad' rectangle B_{i_1} from the proper family and replaced it with a finite collection of rectangles that do not have any 'bad' intersections.

Second, we claim that for all k and any $j \neq i_1, i_2, \ldots, i_n$, we have that $\phi_{s_k} R_k \notin F(B_j; \mathcal{B}^1, \mathcal{D}^1)$. This is true for the following reason. Since $j \neq i_l$, $F(B_j; \mathcal{B}, \mathcal{D}) = \varnothing$. Therefore $B_{i_1} \notin F(B_j; \mathcal{B}, \mathcal{D})$ prior to refining and replacing B_{i_1}. Therefore, either $B_j \cap \phi_{[-2\alpha,2\alpha]} B_{i_1} = \varnothing$ or $B_j \subseteq \phi_{(-3\alpha,3\alpha)} D_{i_1}$. Either condition is 'open,' in the sense that there is some $\epsilon_3(j) > 0$ such that the condition remains true if (B_{i_1}, D_{i_1}) is replaced by $(\phi_s B_{i_1}, \phi_s D_{i_1})$ for $|s| < \epsilon_3(j)$. Therefore, if we further demand that all s_k satisfy $|s_k| < \min_j \epsilon_3(j)$, we will have that $\phi_{s_k} R_k \notin F(B_j; \mathcal{B}^1, \mathcal{D}^1)$, as desired. This implies that $F(B_j; \mathcal{B}^1, \mathcal{D}^1) = \varnothing$ for all such j.

From these two facts we conclude that the set of $B \in \mathcal{B}^1$ for which $F(B; \mathcal{B}^1, \mathcal{D}^1) \neq \varnothing$ is (at most) $B_{i_2}, \ldots B_{i_n}$. To complete the proof, we carry out the 'refine-and-replace' scheme finitely many times, modifying $(\mathcal{B}^1, \mathcal{D}^1)$ into $(\mathcal{B}^2, \mathcal{D}^2)$ by carrying out the procedure above on (B_{i_2}, D_{i_2}), etc. Finally, we modify (B_{i_n}, D_{i_n}) to produce a collection $(\mathcal{B}^n, \mathcal{D}^n)$ which by construction satisfies $F(B; \mathcal{B}^n, \mathcal{D}^n) = \varnothing$ for all $B \in \mathcal{B}^n$. In other words, we have eliminated every intersection which causes the pre-Markov property to fail, and this completes the proof. $\square$

For the intermediate family $\mathcal{K}$ that appears in the construction of Markov proper families, we choose closed rectangles K_i so $K_i \subset \operatorname{Int} B_i$. They can be chosen as close to B_i as we like so that $\{\phi_{[-\alpha,0]}(\operatorname{Int} K_1), \ldots, \phi_{[-\alpha,0]}(\operatorname{Int} K_n)\}$ is an open cover. Now take a Lebesgue number 12δ for this open cover. Then for any x, $\overline{B}(x, 6\delta) \subset \phi_{[-\alpha,0]}(\operatorname{Int} K_i)$ for some i, and thus $\overline{B}(x, 6\delta) \subset \phi_{[-2\alpha,2\alpha]} K_i$. We now prove Proposition 4.10 by showing that we can ensure the sections B_i are geometrically defined rectangles.

Proof of Proposition 4.10. We show that we can construct a proper family out of good geometric rectangles. Let α be small enough that all $\{g_t\}$-orbits of length 8α remain local. Fix $\rho > 0$ much smaller than α. Fix some large τ and for each $c \in G\tilde{X}$ pick an open good geometric rectangle $\tilde{R}(c, \tau)$ with diameter less than ρ. Then $\{g_{(-\rho,0)} \tilde{R}(c, \tau)\}_{c \in G\tilde{X}}$ is an open cover of $G\tilde{X}$. By compactness of GX, we can choose a finite set $\{\tilde{H}_1, \ldots, \tilde{H}_n\}$, writing $\tilde{H}_i = g_{(-\rho,0)} \tilde{R}_i$, so that GX is covered by the projections $H_i = g_{(-\rho,0)} R_i$ of $\tilde{H}_i$ to GX. We build our proper family recursively. Let $B_1 \subset D_1$ be a closed good geometric rectangle of diameter less than α chosen so that $R_1 \subset \operatorname{Int} B_1$. Note that $H_1 \subset g_{(-\alpha,0)} \operatorname{Int} B_1$.

Now suppose that $\{(B_j, D_j)\}_{j=1}^{l}$ have been chosen satisfying $\operatorname{diam} D_j < \alpha$, $D_j \cap D_k = \varnothing$ for $j \neq k$, and so that each D_j has the form $g_{s_j} R_i$ for some s_j with $|s_i| << \alpha$ and some good geometric rectangle R_i. Let H_i be the element of our cover of smallest index such that $H_i \not\subset \bigcup_{j=1}^{l} g_{[-\alpha,0]} \operatorname{Int} B_j$. We want to build further

(B_j, D_j) covering H_i. Let

$$M_i = \left\{ c \in R_i : g_{(0,\rho)}c \ \cap \ \Big(\bigcup_{j=1}^{l} \operatorname{Int} B_j \Big) = \varnothing \right\} = R_i \smallsetminus \left(\bigcup_{j=1}^{l} g_{(-\rho,0)} \operatorname{Int} B_j \right).$$

M_i is an closed subset of R_i. Pick $\epsilon << \frac{\rho}{4Ll}$, where L is given by Lemma 2.4. By passing to endpoints of its geodesics, M_i can be identified with a closed subset of $U^- \times U^+$, so we can find a finite set $T_1, \ldots, T_n$ of closed rectangles with each T_k identified with some $V_k^- \times V_k^+ \subset U^- \times U^+$ such that $\{\operatorname{Int} T_k\}_{k=1}^n$ cover M_i, $T_k \cap M_i \neq \varnothing$, and $\operatorname{diam} T_k < \epsilon$.

By Lemma 4.7, if for some $t \in [0,\rho]$, $g_t T_k \cap D_j \neq \varnothing$, then for $|t' - t| > 2L\epsilon$, $g_{t'} T_k$ and D_j are disjoint. Since $4L\epsilon << \frac{\rho}{l}$, and since there are at most l of the D_j's which can intersect $g_{[0,\rho]} T_k$, we can pick distinct $s_k \in (0,\rho)$ so that $g_{s_k} T_k \cap (\bigcup_{j=1}^{l} D_j) = \varnothing$.

We add the collection $\{D_k := g_{s_k} T^k\}_{k=1}^n$ to our collection $\{D_j\}$. Inside each new D_k, we choose a slightly smaller closed rectangle B_k so that $\{g_{-s_k} \operatorname{Int} B_k\}$ cover M_i. It is then clear since $\rho < \alpha$ that $\bigcup_{j=1}^{l'} g_{(-\alpha,0)} \operatorname{Int} B_j$ covers H_i.

We continue this way until GX is covered by $\{g_{(-\alpha,0)} \operatorname{Int} B_j\}$ and check the conditions of Definitions 3.5 and 3.7. We have ensured that 3.5(2) is satisfied. Using the Lipschitz property of the flow and the fact that $\epsilon << \alpha$ we can ensure that $\operatorname{diam} D_j < \alpha$ for all j, ensuring condition 3.5(1). We have also ensured 3.5(3) by constructing the D_j disjoint and picking α so small that all orbit segments with length 8α are local. Applying Lemma 4.11 produces a pre-Markov proper family satisfying Definition 3.7. By construction, each D_i in $\mathcal{D}$ is the image of a good geometric rectangle under the flow for a small time. $\square$

We now complete the proof of Theorem B. The flow is a metric Anosov flow by Theorem 3.4. The flow is Hölder by Lemma 2.5. We take a pre-Markov proper family for the flow for which the family of sections $\mathcal{D}$ are good geometric rectangles flowed for some short constant amount of time, as provided by Proposition 4.10. By Propositions 4.8 and 4.9 the return time map and projection map to these sections are Hölder. Thus, we have met the hypotheses of Theorem A and we conclude that the geodesic flow has a strong Markov coding.

5. Projective Anosov representations

We show that the methods introduced in the previous section can be adapted to the geodesic flow $(\mathsf{U}_\rho\Gamma, \{\phi_t\})$ for a projective Anosov representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$, proving Theorem C. This flow is a Hölder continuous topologically transitive metric Anosov flow [BCLS15, Proposition 5.1], so to meet the hypotheses of Theorem A it remains to show there is a pre-Markov proper family of sections to the flow such that the return time function between any two sections is Hölder, and the projection from a flow neighborhood of a section to the section are Hölder. We sketch the proof by showing how to set up analogues of all the objects defined in §4. This will demonstrate that the proof in §4 applies in this setting.

Following [BCLS15], we define the geodesic flow for a projective Anosov representation. Let Γ be a Gromov hyperbolic group. We write $\widetilde{\mathsf{U}_0\Gamma} = \partial_\infty\Gamma^{(2)} \times \mathbb{R}$, and $\mathsf{U}_0\Gamma$ for the quotient $\widetilde{\mathsf{U}_0\Gamma}/\Gamma$. The Gromov geodesic flow (see Champetier [Cha94] and Mineyev [Min05]) can be identified with the $\mathbb{R}$-action on $\mathsf{U}_0\Gamma$.

Definition 5.1. A representation $\rho : \Gamma \to \mathsf{SL}_m(\mathbb{R})$ is a *projective Anosov representation* if:

- ρ has transverse projective limit maps. That is, there exist ρ-equivariant, continuous maps $\xi : \partial^\infty\Gamma \to \mathbb{RP}(m)$ and $\theta : \partial^\infty\Gamma \to \mathbb{RP}(m)^*$ such that if $x \neq y$, then
$$\xi(x) \oplus \theta(y) = \mathbb{R}^m.$$
Here we have identified $\mathbb{RP}(m)^*$ with the Grassmannian of $m-1$-planes in $\mathbb{R}^m$ by identifying $v \in \mathbb{RP}(m)^*$ with its kernel.
- We have the following contraction property (see §2.1 of [BCLS15]). Let $E_\rho = \widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^m/\Gamma$ be the flat bundle associated to ρ over the geodesic flow for the word hyperbolic group on $\mathsf{U}_0\Gamma$, and let $E_\rho = \Xi \oplus \Theta$ be the splitting induced by the transverse projective limit maps ξ and θ. Let $\{\tilde{\psi}_t\}$ be the flow on $\widetilde{\mathsf{U}_0\Gamma} \times \mathbb{R}^m$ obtained by lifting the Gromov geodesic flow on $\mathsf{U}_0\Gamma$ and acting trivially on the $\mathbb{R}^m$ factor. This flow descends to a flow $\{\psi_t\}$ on E_ρ. We ask that there exists $t_0 > 0$ such that for all $Z \in \mathsf{U}_0\Gamma$, $v \in \Xi_Z \setminus \{0\}$ and $w \in \Theta_Z \setminus \{0\}$, we have
$$\frac{\|\psi_{t_0}(v)\|}{\|\psi_{t_0}(w)\|} \leq \frac{1}{2}\frac{\|v\|}{\|w\|}.$$

For $v \in (\mathbb{R}^m)^*$ and $u \in \mathbb{R}^m$, we write $\langle v|u\rangle$ for $v(u)$. We define the geodesic flow $(\mathsf{U}_\rho\Gamma, \{\phi_t\})$ of a projective Anosov representation, referring to §4 of [BCLS15] for further details. Let
$$F_\rho = \left\{(x, y, (u, v)) : (x, y) \in \partial^\infty\Gamma^{(2)},\ u \in \xi(x),\ v \in \theta(y),\ \langle v|u\rangle = 1\right\} / \sim$$
where $(u, v) \sim (-u, -v)$ and $\partial^\infty\Gamma^{(2)}$ denotes the set of distinct pairs of points in $\partial^\infty\Gamma$. Since u determines v, F_ρ is an $\mathbb{R}$-bundle over $\partial^\infty\Gamma^{(2)}$. The flow is given by
$$\phi_t(x, y, (u, v)) = (x, y, (e^t u, e^{-t} v)).$$
We define $\mathsf{U}_\rho\Gamma = F_\rho/\Gamma$. The space $\mathsf{U}_\rho\Gamma$ is compact [BCLS15, Proposition 4.1] (even though Γ does not need to be the fundamental group of a closed manifold). The flow $\{\phi_t\}$ descends to a flow on $\mathsf{U}_\rho\Gamma$. The flow $(\mathsf{U}_\rho\Gamma, \{\phi_t\})$ is what we call the *geodesic flow of the projective Anosov representation.* The flow is Hölder orbit equivalent to the Gromov geodesic flow on $\mathsf{U}_0\Gamma$, which motivates this terminology. In [BCLS15, Theorem 1.10], it is proven that $(\mathsf{U}_\rho\Gamma, \{\phi_t\})$ is metric Anosov. We construct sections locally on F_ρ and project the resulting sections down to $\mathsf{U}_\rho\Gamma$ that will verify the hypotheses of Theorem A, and thus show that the geodesic flow has a strong Markov coding.

We can define stable and unstable foliations in the space F_ρ. For a point $Z = (x_0, y_0, (u_0, v_0)) \in F_\rho$, we define respectively, the strong unstable, unstable, strong stable, and stable leafs through Z as follows.

$$W^{uu}(Z) = \{(x, y_0, (u, v_0)) : x \in \partial^\infty\Gamma, x \neq y_0, u \in \xi(x), \langle v_0|u\rangle = 1\}.$$
$$W^{u}(Z) = \{(x, y_0, (u, v)) : x \in \partial^\infty\Gamma, x \neq y_0, u \in \xi(x), v \in \theta(y_0), \langle v|u\rangle = 1\} = \bigcup_{t\in\mathbb{R}} \phi_t(\mathcal{L}_Z^-).$$
$$W^{ss}(Z) = \{(x_0, y, (u_0, v)) : y \in \partial^\infty\Gamma, x_0 \neq y, v \in \theta(y), \langle v|u_0\rangle = 1\}.$$
$$W^{s}(Z) = \{(x_0, y, (u, v)) : y \in \partial^\infty\Gamma, x_0 \neq y, u \in \xi(x_0), v \in \theta(y), \langle v|u\rangle = 1\} = \bigcup_{t\in\mathbb{R}} \phi_t(\mathcal{L}_Z^+).$$

Fix any Euclidean metric $|\cdot|$ on $\mathbb{R}^m$. This induces a metric on
$$\mathbb{RP}(m) \times \mathbb{RP}(m)^* \times ((\mathbb{R}^m \times (\mathbb{R}^m)^*)/\pm 1).$$

Let d_{F_ρ} be the pull-back of this metric to F_ρ; the transversality condition on the limit maps in the definition of Anosov projective representation ensures this is well-defined. This is called a *linear metric* on F_ρ. There is a Γ-invariant metric d_0 on F_ρ which is locally bi-Lipschitz to any linear metric by [BCLS15, Lemma 5.2]. Therefore, it is sufficient to verify the Hölder properties we want with respect to a linear metric.

We now build our sections, by analogy with our construction of good geometric rectangles in the CAT(−1) setting. Fix some $Z = (x_0, y_0, (u_0, v_0)) \in F_\rho$ and choose some small, disjoint open sets U^+ containing x_0 and U^- containing y_0. Choose U^+ and U^- small enough that for all $x, y \in U^+ \times U^-$, $\xi(x)$ and $\theta(y)$ are transversal. Since $\xi(x_0)$ and $\theta(y_0)$ are transversal and ξ, θ are continuous, this is possible. Let

$$R(Z, U^+, U^-) = \{(x, y, (u, v)) \in \mathsf{U}_\rho\Gamma : x \in U^+, y \in U^-, \langle v_0 | u\rangle = 1\}.$$

It is straightforward to check that $R(Z, U^+, U^-)$ is a transversal to the flow ϕ_t by using the definition of a linear metric to verify that all points sufficiently near to Z project to $R(Z, U^+, U^-)$. It is also straightforward to check that $R(Z, U^+, U^-)$ is a rectangle using the definitions of the (strong) stable and unstable leaves. This is essentially the same as our proof of Lemma 4.3. We can describe $R(Z, U^+, U^-)$ as the zero set for a 'Busemann function' as follows.

Lemma 5.2. *Fix $Z_0 = (x_0, y_0, (u_0, v_0))$. For all $(x, y, (u, v))$ define*

$$\beta_{Z_0}((x, y, (u, v))) = -\log\langle v_0 | u\rangle.$$

Then β_{Z_0} is a locally Lipschitz function with respect to a linear metric on F_ρ.

Proof. Let $Z_1 = (x_1, y_1, (u_1, v_1))$ and $Z_2 = (x_2, y_2, (u_2, v_2))$ be in a small neighborhood of Z_0 for the linear metric. This implies that $\langle v_0, u_i\rangle$ lie in some range bounded away from zero. Over this range, the function $-\log$ is Lipschitz.

We know by the definition of a linear metric that

$$d_{F_\rho}(Z_1, Z_2) = |\xi(x_1) - \xi(x_2)| + |\theta(y_1) - \theta(y_2)| + |u_1 - u_2| + |v_1 - v_2|.$$

(In the various factors above, $|* - *|$ denotes the metrics induced on $\mathbb{RP}(m)$, $\mathbb{RP}(m)^*$, $\mathbb{R}^m$, and $(\mathbb{R}^m)^*$ but the Euclidean metric on $\mathbb{R}^m$.) We calculate, using that $-\log$ and $\langle v_0 | \cdot\rangle$ are Lipschitz:

$$\begin{aligned} |\beta_{Z_0}(Z_1) - \beta_{Z_0}(Z_2)| = |-\log\langle v_0|u_2\rangle - \log\langle v_0|u_1\rangle| &\le K_1 |\langle v_0|u_2\rangle - \langle v_0|u_1\rangle| \\ &\le K_2 |u_2 - u_1| \\ &\le K_2 d_{F_\rho}(Z_1, Z_2). \end{aligned}$$ □

Lemma 5.3. *For all $Z \in R(Z_0, U^+, U^-)$, we have $\beta_{Z_0}(\phi_t Z) = -t$.*

Proof. This is immediate from the definition of β_{Z_0}. □

It is clear that $R(Z_0, U^+, U^-) = \{(x, y, (u, v)) : x \in U^+, y \in U^-, \beta_{Z_0}(u) = 0\}$ and if $\phi_{t^*} Z \in R(Z_0, U^+, U^-)$, then $\beta_{Z_0}(Z) = t^*$. We now have a simple proof of the analogue of Proposition 4.8 we need:

Proposition 5.4. *The return time function between two good geometric rectangles is Lipschitz.*

Proof. Suppose that $Z_1, Z_2 \in R$ and, for small r_1, r_2, that $\phi_{r_1} Z_1, \phi_{r_2} Z_2 \in R' = R(Z', U'^+, U'^-)$. Then by Lemma 5.2, we have

$$|r_1 - r_2| = |\beta_{Z'}(Z_1) - \beta_{Z'}(Z_2)| \le K d_{F_\rho}(Z_1, Z_2)$$ □

It is also easy to verify that the flow $\{\phi_t\}$ is Lipschitz. All that is left to prove is an analogue of Proposition 4.9:

Lemma 5.5. *For any good geometric rectangle R, $\mathrm{Proj}_R : \phi_{(-\alpha,\alpha)}R \to R$ is Hölder.*

Proof. Since the flow is Lipschitz, we can assume $Z_1 \in R$. $Z_2 \in \phi_{-t^*}R$ for some $t^* \in (-\alpha, \alpha)$, and $\mathrm{Proj}_R(Z_2) = \phi_{t^*} Z_2$. If R is a rectangle based at $Z_0 = (x_0, y_0, (u_0, v_0))$, then $(u_2, v_2) \mapsto (e^{t^*}u_2, e^{-t^*}v_2)$ is the projection along the smooth flow (e^t, e^{-t}) to the smooth subset of $\mathbb{R}^m \times (\mathbb{R}^m)^*$ given by $\{(u,v) : \langle v_0 | u\rangle = 1, \langle v | u\rangle = 1\}$, which is transverse to the flow. Therefore this map is smooth, hence Lipschitz on any compact set for any linear metric, and this suffices for the proof. □

6. Applications of strong markov coding

There is a wealth of literature for Anosov and Axiom A flows which uses the strong Markov coding to prove strong dynamical properties of equilibrium states. We do not attempt to create an exhaustive list of these applications, but we refer the reader to the many results described in references such as Bowen-Ruelle [BR75], Pollicott [Pol87], Denker-Philipp [DP84] and Melbourne-Török [MT04]. We note in particular the many useful properties that follow from the existence of a strong Markov coding that are used in [Sam16, BCLS15], etc.

We summarize some of these applications as they apply to the geodesic flow of a compact locally CAT(-1) space $X = \tilde{X}/\Gamma$. The flow is topologically transitive since the action of Γ on $\partial^\infty\Gamma^{(2)}$ is topologically transitive. (For a general metric Anosov flow, the now-wandering set satisfies the Smale spectral decomposition theorem into transitive components [Pol87, Theorem 4].) In places in the discussion below, we need the notion of topological weak-mixing. We say that a metric Anosov flow is *topologically weak-mixing* if all closed orbit periods are not integer multiples of a common constant.

The result that there is a unique equilibrium state μ_φ for every Hölder potential is due to Bowen-Ruelle [BR75] for topologically transitive Axiom A flows. The method of proof was observed to extend to flows with strong Markov coding in [Pol87]. It is also observed in [Pol87] that if φ, ψ are Hölder continuous functions then the map $t \to P(\varphi + t\psi)$ is analytic and $(d/dt)P(\varphi + t\psi)|_{t=0} = \int \psi d_{\mu_\varphi}$, where $P(\cdot)$ is the topological pressure. This result is one of the key applications of thermodynamic formalism used in [BCLS15].

We now discuss the statistical properties listed in (1) of Corollary D. The Almost Sure Invariance Principle (ASIP), Central Limit Theorem (CLT), and Law of the Iterated Logarithm are all properties of a measure that are preserved by the push forward π^* provided by the strong Markov coding, and thus it suffices to establish them on the suspension flow. The CLT is probably the best known of these results, and goes back to Ratner [Rat73]. A convenient way to obtain these results in our setting is to apply the paper of Melbourne and Török [MT04] which gives a relatively simple argument that the CLT lifts from an ergodic measure in the base to the corresponding measure on the suspension flow. They than carry out the more difficult proof that the ASIP lifts from an ergodic measure in the base to the flow, recovering the result of Denker and Phillip [DP84]. The other properties discussed (and more, see [MT04]), are a corollary of ASIP. The equilibrium state for the suspension flow is the lift of a Gibbs measure on a Markov shift. The measure in the base therefore satisfies ASIP by [DP84], so we are done.

We now discuss the application to dynamical zeta functions, which is claimed in the case there is a strong Markov coding and the flow is topologically weak mixing in [Pol87]. Results on zeta functions are carried over from the suspension flow by a strong Markov coding. The assumption of topological weak mixing is not needed for the result that we stated as (2) in Corollary D. See [PP90, Chapter 6] for a discussion of how the topologically weak-mixing property impacts other properties of the zeta function.

For item (3) of Corollary D, we can refer directly to [Pol87] for the statement that if the flow has a strong Markov coding and is topological weak-mixing, then the equilibrium state μ_φ is Bernoulli. The proof is given by Ratner [Rat74].

For item (4) of Corollary D , we argue as follows. Ricks proves that for a proper, geodesically complete, CAT(0) space $\tilde{X}$ with a properly discontinuous, cocompact action by isometries Γ, all closed geodesics have lengths in $c\mathbb{Z}$ for some $c > 0$ if and only if $\tilde{X}$ is a tree with all edge lengths in $c\mathbb{Z}$ [Ric17, Thorem 4]. It follows that X is a metric graph with all edges of length c. In this case, the symbolic coding for the geodesic flow on X is explicit: (GX, g_t) is conjugate to (Σ_A^c, ϕ_t), the suspension flow with constant roof function c over the subshift of finite type defined by the adjacency matrix A for the graph X. Equilibrium states for the flow are products of equilibrium states in the base with Lebesgue measure in the flow direction. Since an equilibrium state for a Hölder potential on a topologically mixing shift of finite type is Bernoulli, item (4) follows immediately by taking $k \geq 1$ so that A^k is aperiodic; if $k = 1$, the measure on the base is Bernoulli, and if $k > 1$ the measure on the base is the product of Bernoulli measure and rotation of a finite set with k elements.

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